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A STEADY LAMINAR FLOW OF A SUSPENSION IN A CHANNEL

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Abstract—A two dimensional model of a steady flow of a suspension in an inclined channel is developed and studied with the aid of an integral method. The model explains the existence of two operational modes in such a channel, predicted earlier by Probstein *et al.* (1977).

INTRODUCTION

The lamella settler (LS) is a clarifier consisting of parallel channels, formed by a number of inclined plates, stacked as shown schematically in figure 1(a). The slurry, containing the solid particles to be clarified, is fed into the channels. The sludge consisting of the settled particles slides down the bottom wall of each channel and is collected through small orifices in the bottom ends of the channels. Meanwhile the clarified liquid, "squeezed" up by the heavy column of suspension, appears at the top fo the channels and is removed.

Probstein *et al.* (1977) have suggested a locally one-dimensional model (referred to below as Model I) for describing the dynamics of lamella settlers. An analysis based on Model I disclosed the possibility of the existence of two alternative operational regimes of LS for a

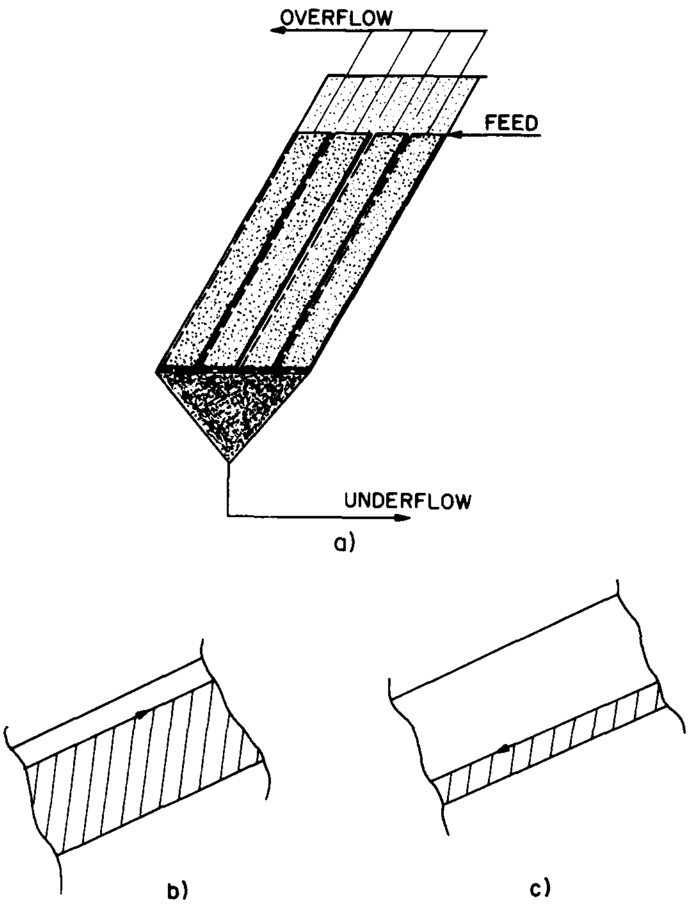

Figure 1. (a) Schematic diagram of lamella settler. The (b) subcritical and (c) supercritical retimes.

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given clear water overflow: the subcritical and supercritical operational modes. In the subcritical regime the suspension occupies most of the channel and the interface separating the clear water from the suspension moves up the channel (see figure 1b). In the supercritical regime an opposite situation takes place (figure 1(c)).

The existence of two alternative operational modes, derived from Model I and observed experimentally (Probstein & Hicks, 1978) is of potential importance for LS technology. (There are reasons to believe that the supercritical mode, which has not been exploited in LS practice, possesses some important advantages as compared to the subcritical one.)

Model I was built upon the following essential assumptions:

(1) It was assumed that there are three layers of viscous, immiscible fluids present in the channel: clear water, suspension and mud.

(2) The movement of the liquid layers was treated as a steady, laminar, fully developed gravity flow between parallel inclined planes. (The bottom end of the channel was assumed closed so that the net volumetric flow through the channel was taken equal to zero.)

In addition to these assumptions several other less essential ones were introduced.

The existence of two alternative operating modes came out of Model I as a nonuniqueness of a numerical solution of a set of algebraic equations describing the position of the interfaces between the liquid layers and the coefficients of the corresponding parabolic velocity profiles as functions of the "clear water" flow. The nature of these two regimes remained unclear. Moreover, the essentially one-dimensional approach of Model I left unclear the connection between the conditions at the entrance to the channel (such as the initial thickness of the layer occupied by the suspension, etc.) and the asymptotic establishment of one or the other of the alternative operational regimes far from the entrance to the channel. The present paper investigates by a multi-phase two dimensional flow model the approach to LS dynamics suggested by Probstein *et al.* (1977) and in particular clarifies the nature of two above mentioned alternative operational regimes.

The aim of section 1 is to establish a connection between the approach adopted in Model I, as represented by assumption 1, and the standard methods of the theory of two-phase flows. To this end we formulate in section 1 the boundary value problem describing a three dimensional two-phase flow of a suspension in a channel. It is shown that this boundary value problem can be reduced approximately to one of a gravitational flow of two immiscible fluids with a macroscopic interface, whose position has to be determined. In section 2 a simplified two-dimensional model is analyzed based, as in Model I, on the assumption of immiscibility of the liquids flowing in the channel. This model, investigated with the aid of a simple integral method, explains the existence of the alternative operational modes. Furthermore, it enables one to relate the realization of a particular mode to conditions at the entrance to the channel.

1. EQUATIONS OF STEADY, LAMINAR TWO-PHASE FLOW OF A SUSPENSION IN A CHANNEL

A detailed account of questions related to the formulation of the equations of two-phase flow can be found in Ishii (1975), Soo (1967) or Pai (1977). We emphasize that the form chosen below for the equations is by no means unique, but rather one of several equivalent forms (e.g. Ishii, 1975). Consider a steady laminar flow in a channel of a suspension composed of a viscous incompressible fluid and identical solid particles suspended in it. We suppose the particles to be large enough to make their Brownian diffusion negligibe. Introduce the notation: ρ_1—density of a particle; c—volume fraction of particles; ρ_0—density of the liquid; $\mathbf{V}_1$—average local velocity of the particles; and $\mathbf{V}_0$—average local velocity of the liquid.

We define:

The average density of the suspension ρ

$$\rho = c\rho_1 + (1-c)\rho_0 . \qquad [1.1]$$

The average volumetric velocity of the mixture **W**

$$\mathbf{W} = cV_1 + (1-c)\mathbf{V}_0. \quad [1.2]$$

The average mass velocity of the mixture **U**

$$\mathbf{U} = [c\rho_1\mathbf{V}_1 + (1-c)\rho_0\mathbf{V}_0]/\rho. \quad [1.3]$$

The conservation laws used in what follows are:

Conservation of mass

$$\operatorname{div}(\rho_k c_k \mathbf{V}_k) = 0 \quad k = 0, 1. \quad [1.4]$$

Conservation of momentum

The momentum balance equation for the mixture under consideration is of the form (see Pai 1977)

$$\rho(\mathbf{U}\cdot\nabla)\mathbf{U} = -\nabla P + \rho\mathbf{g} + \nabla\cdot\boldsymbol{\sigma}. \quad [1.5]$$

Here we have introduced the notation: P—pressure; **g**—gravitational acceleration; and $\boldsymbol{\sigma}$—average deviatoric stress tensor.

Equation [1.5] has generally to be supplemented with another momentum balance equation for one of the mixture's components. Though, we assume, as is often done (e.g. Haase 1969, Hill *et al.* 1977, Acrivos & Herbolzheimer 1979), a complete local relaxation between the momenta of the different components of the mixture. A common version of this assumption is the slip hypothesis used here, of the form:

$$\mathbf{V}_1 = \mathbf{W} + \beta f(c)\cdot\mathbf{g}. \quad [1.6]$$

Here $f(c)$ is a function of the volume fraction of the particles which characterizes their interaction, and β is a constant. For the simplest case of noninteracting spherical particles $(f(c)\equiv 1)\beta$ is determined by Stokes formula

$$\beta_0 = \frac{2}{9}\frac{\rho_1-\rho_0}{\mu}R_0^2 \quad [1.7]$$

where μ is the viscosity of the liquid and R_0 is the radius of the particle.

The system [1.1]–[1.8] has to be supplemented with a constitutive relation for $\boldsymbol{\sigma}$. We content ourself with the simplest possible choice (see Pai 1977)

$$\sigma_{ij} = \mu(c)\left[\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} - \frac{2}{3}\frac{\partial U_k}{\partial x_k}\delta_{ij}\right]. \quad [1.8]$$

Here $\mu(c)$ is the effective viscosity of the suspension, δ_{ij} is a unit tensor and a summation is carried out over a repeated subscript.

The formulation of the basic equations is complete and we rewrite them below in a form convenient for future use. From [1.5] and [1.8]

$$\rho(\mathbf{U}\cdot\nabla)\mathbf{U} = -\nabla P + \rho\mathbf{g} + \mu\Delta\mathbf{U} + \frac{1}{3}\mu\nabla(\nabla\mathbf{U}). \quad [1.9]$$

Furthermore, from [1.1], [1.4]

$$\operatorname{div} \mathbf{U} + \Delta\rho \operatorname{div} c\mathbf{U} = 0 \qquad [1.10a]$$

where

$$\Delta\rho = \rho_1 - \rho_0 . \qquad [1.10b]$$

Using [1.6], [1.2] and [1.3] one obtains from [1.4] with $k = 1$

$$\operatorname{div} c\mathbf{U} + \operatorname{div}[F(c)\beta\mathbf{g}] = 0 \qquad [1.11a]$$

where

$$F(c) = \frac{c(1-c)\rho_0}{c\rho_1 + (1-c)\rho_0} f(c) . \qquad [1.11b]$$

Equations [1.9]–[1.11] fully describe the flow of a suspension within the adopted approximation of a purely convective, locally relaxed flow.

It should be noted that from [1.4] and [1.11] it follows that the volume fraction of the particles is conserved along their trajectories. In other words, if the particle concentration (volume fraction) is constant and equal to c_0 at the entrance to the channel (an assumption made everywhere below) it will remain so in every place where the particles are brought by the flow. Thus the whole channel can be considered divided into two regions: one in which the volume fraction of the particles is constant and equal to c_0 (the suspension) and the other one into which no particles arrive (the clear water). These two regions are separated by a "limiting particle trajectory" representing an example of a kinematic shock common in the theory of two-phase flows (Ishii 1975). It is the inevitability of the formation of such a shock that is responsible for the enhancement of sedimentation in an inclined channel, known as the Boycott effect (Hill *et al.* 1977, Acrivos & Herbolzheimer 1979).

As described, the flow of suspension is reduced to a purely hydrodynamical problem of the flow of a viscous liquid with a piecewise constant density and with an unknown location of the surface of discontinuity. In the regions of constant density the flow is described by [1.9]–[1.11] with

$$c^{(2)} = c_0, \quad \rho^{(2)} = c_0\rho_1 + (1-c_0)\rho_0 \qquad [1.12a]$$

and

$$c^{(1)} = 0, \rho^{(1)} = \rho_0 \qquad [1.12b]$$

where the superscripts 1, 2 refer to the "clear water" and "suspension" respectively. For the sake of simplicity of presentation we have excluded from the consideration the thin mud layer adjacent to the bottom of the channel made up of the sedimented particles. For the same reason we will restrict ourself in the following sections to considering the flow in a vertical channel. It was pointed out by Probstein *et al.* (1977) that the presence of the thin layer has little influence on the dynamics of the flow as a whole.

Equations [1.9]–[1.12] must be supplemented with conditions at the unknown surface of density discontinuity, S, namely:

(1) *Kinematic condition*

$$\mathbf{V}_i \cdot \mathbf{n}|_S = 0 \qquad [1.13]$$

which expresses the fact that there are no particles crossing the surface of the density discontinuity. Here $\mathbf{n}$ is normal to S and $\mathbf{V}_1$ is the particle velocity.

(2) *Condition of continuity of the tangential velocity component*

$$[\mathbf{U}_t]_S = 0 \quad [1.14a]$$

where [ ] denotes a jump.†

(3) *Condition of local equilibrium of the interface*

$$[\boldsymbol{\sigma} \cdot \mathbf{n}]_S = 0 . \quad [1.15]$$

We have assumed here that the surface tension of the discontinuity surface S is equal to zero.

The system [1.9]–[1.15] together with [1.3], [1.6] and [1.8] supplemented by the boundary conditions at the fixed boundaries represents a boundary value problem with respect to the position of the surface of density discontinuity and the fields of pressure and mean mass velocity. In the following sections the system will be studied in detail for a particular example. However, we point out here that according to [1.2], [1.3] and [1.6]

$$\mathbf{V}_1 = \mathbf{U} + \mathbf{V}_f \quad [1.16a]$$

where

$$\mathbf{V}_f = \frac{(1-c)\rho_0}{c\rho_1 + (1-c)\rho_0} f(c)\beta \mathbf{g} . \quad [1.16b]$$

For a typical particle $\rho_1 \sim 2000\, \mathrm{kg/m^3}$, $\mu \sim 10^{-3}\, \mathrm{kg/ms}$, so that $V_f \sim 10^{-5}\, \mathrm{m/s}$. Thus, in those cases where the mass velocity component in the $\mathbf{g}$ direction at the surface of discontinuity is much larger than this value, it appears natural to neglect the terms V_f as compared to $\mathbf{U} \cdot \mathbf{g}/g$. A "limiting" particle trajectory then becomes a "limiting" mass velocity streamline, and the flow in the channel can be treated as one of immiscible fluids. This assumption will be adopted in what follows. Its admissibility should, of course, be checked *a posteriori* by comparing the calculated values of $\mathbf{U} \cdot \mathbf{g}/g|_S$ with V_f.

2. A STEADY FLOW OF IMMISCIBLE FLUIDS IN A VERTICAL CHANNEL

2.1 *The basic boundary value problem*

Let us consider a semi-infinite vertical channel of width h, closed at infinity, and in which there is a steady two-dimensional flow of a suspension. The volume flux of suspension $\tilde{Q}$ is fed into the channel through an orifice occupying a specified fraction of the channel's width. (We here denote dimensional variables with the overbar "$\tilde{}$".) The volume fraction of particles c_0 in the suspension is assumed constant (see figure 2). We assume that solid particles are removed instantaneously from the system at infinity. Thus, at steady state, the channel can be considered divided into two regions: one occupied by suspension with a constant density $\rho = c_0\rho_1 + (1-c_0)\rho_0$ and another of "clear" water with density $\rho_0 < \rho$, separated by a density discontinuity interface $\tilde{\delta}(x)$ (the axes are directed as shown in figure 2). The suspension and "clear" water are treated as immiscible viscous fluids in the sense of section 1. The channel is assumed vertical in order to exclude from consideration the complications connected with particle deposition on the walls. (The angle of inclination of the channel is of crucial importance

†From [1.10], [1.11] the condition for the jump in normal velocity is

$$[\mathbf{U}_n]_S = \frac{\Delta\rho c(1-c)\rho_0}{c\rho_1 + (1-c)\rho_0} f(c)\mathbf{gn} . \quad [1.14b]$$

It can be seen that fulfillment of [1.14b] follows from [1.11] and [1.14a].

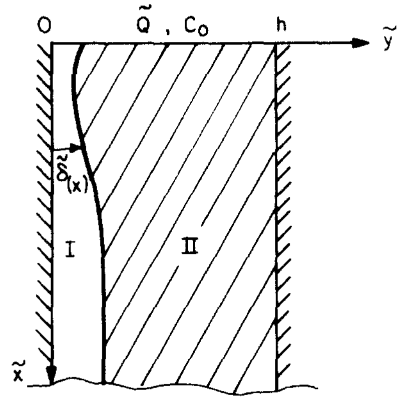


Figure 2. Geometry of flow in a vertical channel.

when considering the stability of the steady flow. Stability questions however, are not considered in the present work.)

We shall begin by considering the flow in creeping approximation (though the Reynolds numbers for practically relevant flows in lamella settlers are in the range $1 < \mathrm{Re} < 500$). We start with the creeping approximation because of its simplicity and the idea that it will lead to qualitatively instructive results. Inertial effects will be considered in section 2.6.

For the stated assumptions the equations of section 1, written in dimensionless variables, become

$$\eta_i \nabla^2 U_i = \frac{\partial P_i}{\partial x} - G_i, \quad i = 1, 2 \qquad [2.1]$$

$$\eta_i \nabla^2 V_i = \frac{\partial P_i}{\partial y}, \quad i = 1, 2 \qquad [2.2]$$

$$\frac{\partial U_i}{\partial x} + \frac{\partial V_i}{\partial y} = 0, \quad i = 1, 2. \qquad [2.3]$$

Here the subscript $i = 1$ refers to region I (see figure 2), occupied by the "clear" water, $i = 2$ refers to the region II occupied by the suspension. The dimensionless longitudinal and transverse coordinates are given by $x = (\tilde{x}/h)$, $y = (\tilde{y}/h)$, velocity components are $U = (\tilde{U}/U_0)$, $V = (\tilde{V}/U_0)$ (the normalizing constant U_0 will be specified later). Furthermore $\eta_1 = 1$, $\eta_2 = (\mu_2/\mu_1)$, $G_i = (\rho_i g h^2/U_0\mu_1)$ while $P_i = (\tilde{P}_i/\mu_1 U_0)$ is the dimensionless pressure.

Taking into account the zero volume flux through the channel condition [1.16] transforms to

$$\int_0^{\delta(x)} U_1 \, \mathrm{d}y = -Q. \qquad [2.4]$$

Here $Q = (\tilde{Q}/U_0 h)$ is the dimensionless volume flux of suspension and $\delta(x) = (\tilde{\delta}(x)/h)$ is the dimensionless thickness of the "clear" water layer. Continuity of mass velocity at the interface leads to

$$U_1|_{y=\delta(x)} = U_2|_{y=\delta(x)} \qquad [2.5]$$

(an analogous equality for V_1, V_2 follows from the continuity equation [2.3]). Condition (1.15) yields

$$\left[2\eta_i\left(\frac{\partial U_i}{\partial x} - \frac{\partial V_i}{\partial y}\right)n_x n_y + \eta_i\left(\frac{\partial U_i}{\partial y} + \frac{\partial V_i}{\partial y}\right)(n_y^2 - n_x^2)\right]_{y=\delta(x)} = 0 \qquad [2.6]$$

for the tangential stress component and

$$\left[-P_i+2\eta_i\left(\frac{\partial U}{\partial x}n_x^2+\frac{\partial V_i}{\partial y}n_y^2\right)+2\eta_i\left(\frac{\partial U_i}{\partial x}+\frac{\partial V_i}{\partial x}\right)n_xn_y\right]_{y=\delta(x)}=0 \qquad [2.7]$$

for the normal stress component. Here

$$n_x=-\frac{\delta'_{(x)}}{\sqrt{(1+\delta'^2)}},\quad n_y=\frac{1}{\sqrt{(1+\delta'^2)}} \qquad [2.8a, b]$$

are the components of the interface normal in the x and y directions, respectively. Finally, the condition of zero volume flux through the channel reads

$$\int_0^{\delta(x)} U_1\,\mathrm{d}y+\int_{\delta(x)}^1 U_2\,\mathrm{d}y=0. \qquad [2.9]$$

Equations [2.1]–[2.9] must be supplemented by the no-slip conditions at the channel walls:

$$U_1|_{y=0}=V_1|_{y=0}=U_2|_{y=1}=V_2|_{y=1}=0. \qquad [2.10]$$

Formulation of the nonlinear boundary value problem [2.1]–2.10] for U_i, V_i, P_i, $\delta(x)$ ($i=1,2$) is thus completed by applying the appropriate entrance conditions at $x=0$ and requiring the solution be bounded as $x\to\infty$.

2.2 *The integral method*

We propose to treat the boundary value problem [2.1]–[2.10] by means of a simple integral method. The procedure to be adopted is as follows.

We approximate the longitudinal velocity components U_i by polynomials of the form

$$U_1(x, y)=A_1(x)y^2+B_1(x)y \qquad [2.11]$$

$$U_2(x, y)=A_2(x)(1-y)^2+B_2(x)(1-y). \qquad [2.12]$$

The profiles [2.11] and [2.12] satisfy the boundary conditions [2.10].

Furthermore, one has from the continuity equation [2.3], taking into account [2.10],

$$V_1=-A_1'(x)\frac{y^3}{3}-B_1'(x)\frac{y^2}{2} \qquad [2.13]$$

and

$$V_2=A_2'\frac{(1-y)^3}{3}+B_2'\frac{(1-y)^2}{2}. \qquad [2.14]$$

Substitution of [2.13] and [2.14] into [2.2] yields after integration

$$P_1(x, y)=P_1^0(x)-\eta_1\left[A_1'''\frac{y^4}{12}+B_1'''\frac{y^3}{6}+A_1'y^2+B_1'y\right] \qquad [2.15]$$

$$P_2(x, y)=P_2^0(x)-\eta_2\left[A_2'''\frac{(1-y)^4}{12}+B_2'''\frac{(1-y)^3}{6}+A_2^1(1-y)^2+B_2^1(1-y).\right. \qquad [2.16]$$

Here $P_1^0(x)$, $P_2^0(x)$ are unknown functions of x alone which have to be determined.

The next step is to require [2.1] to be fulfilled in an integral sense over the regions $0 < y < \delta$ for $i = 1$ and $\delta < y < 1$ for $i = 2$, respectively. Namely, we write

$$\int_0^{\delta(x)} \left[\eta_1\left(\frac{\partial^2 U_1}{\partial x^2} + \frac{\partial^2 U_1}{\partial y^2}\right) - \frac{\partial P_1}{\partial x} + G_1\right] \mathrm{d}y = 0 \qquad [2.17]$$

and

$$\int_{\delta(x)}^1 \left[\eta_2\left(\frac{\partial^2 U_2}{\partial x^2} + \frac{\partial^2 U_2}{\partial y^2}\right) - \frac{\partial P_2}{\partial x} + G_2\right] \mathrm{d}y = 0\,. \qquad [2.18]$$

Substitution of [2.11] and [2.12] into [2.17] and [2.18] yields after integration

$$\eta_1\left[A_1{}^{IV}\frac{\delta^4}{60} + B_1{}^{IV}\frac{\delta^3}{24} + \frac{2}{3}A_1{}^{II}\delta + B_1{}^{II}\delta + 2A_1\right] - P_1{}^{0I} = -G_1 \qquad [2.19]$$

and

$$\eta_2\left[A_2{}^{IV}\frac{(1-\delta)^4}{60} + B_2{}^{IV}\frac{(1-\delta)^3}{24} + \frac{2}{3}A_2{}^{II}(1-\delta)^2 + B_2{}^{II}(1-\delta) + 2A_2\right] - P_2{}^{0I} = -G_2\,. \qquad [2.20]$$

Equations [2.19] and [2.20] are the integral versions of momentum equations [2.1] for the longitudinal velocity components.

Substitution of [2.11] and [2.12] into condition [2.5] yields

$$a_1\delta^2 + B_1\delta = A_2(1-\delta)^2 + B_2(1-\delta)\,. \qquad [2.21]$$

In order to simplify the algebra further, we assume next

$$\mu_1 = \mu_2 = \mu(\eta_2 = 1)\,. \qquad [2.22]$$

With [2.22] in mind we obtain upon substituting [2.11]–[2.16] into conditions [2.6] and [2.7], after some transformations

$$2A_1\delta + B_1 + 2A_2(1-\delta) + B_2 = 0 \qquad [2.23]$$

and

$$-P_1{}^0 + \eta_1\left(A_1{}^{III}\frac{\delta^4}{12} + B_1{}^{III}\frac{\delta^3}{6}\right) = -P_2{}^0 + \eta_2\left(A_2{}^{III}\frac{(1-\delta)^4}{12} + B_2{}^{III}\frac{(1-\delta)^3}{6}\right). \qquad [2.24]$$

Finally, conditions [2.4] and [2.9] transform into

$$A_1\frac{\delta^3}{3} + B_1\frac{\delta^2}{2} = -Q \qquad [2.25]$$

$$A_1\frac{\delta^3}{3} + B_1\frac{\delta^2}{2} + A_2\frac{(1-\delta)^3}{3} + B_2\frac{(1-\delta)^2}{2} = 0\,. \qquad [2.26]$$

On solving [2.21], [2.23], [2.25] and [2.26] for A_i, B_i $(i = 1, 2)$ one obtains

$$A_1 = \frac{3}{2}Q\frac{1+2\delta}{\delta^3(1-\delta)} \qquad [2.27]$$

$$A_2 = -\frac{3}{2}Q\frac{3-2\delta}{\delta(1-\delta)^3} \qquad [2.28]$$

$$B_1 = -3Q\frac{1}{\delta^2(1-\delta)} \qquad [2.29]$$

$$B_2 = 3Q\frac{1}{\delta(1-\delta)^2}. \qquad [2.30]$$

Substracting [2.20] from [2.19] and using [2.24] for expressing $P_2^{01} - P_1^{01}$ leads after some transformations to

$$\frac{1}{30}\left[A_2^{IV}(1-\delta)^5 - A_1^{IV}(1-\delta)\delta^4\right] + \frac{1}{16}\left[B_2^{IV}(1-\delta)^4 - B_1^{IV}\delta^3(1-\delta)\right]$$

$$+ A_1^{II}\frac{\delta^2}{3} + B_1^{II}\frac{\delta}{2} + (A_1 - A_2)(1+\delta'^4)(1-\delta) = (G_2 - G_1)\frac{1-\delta}{2} \qquad [2.31]$$

which reduces [2.19]–[2.26] to a single nonlinear ordinary differential equation for the interface location $\delta(x)$. Before we solve this equation numerically in section 2.4, we shall discuss the asymptotic behavior its bounded solutions (section 2.3). These asymptotic solutions correspond to developed channel flows which are observed in the lamella settlers far enough down from the feeding orifice, and described by the one-dimensional model due to Probstein *et al.*

2.3 *Bounded, monotonic asymptotic behavior of the interface*

One has, for an interface bounded and monotonic far from the entrance to the channel, assuming that all the derivatives in [2.31] tend to zero as $x \to \infty$,

$$A_1^\infty - A_2^\infty = (G_2 - G_1)/2$$

or, using [2.27], [2.28]

$$Q = \frac{G_2 - G_1}{3}\delta_\infty^3(1-\delta_\infty)^3 \qquad [2.32a]$$

and in dimensional form

$$\tilde{Q} = \frac{\Delta\rho g}{3\mu}\frac{\tilde{\delta}_\infty^3}{h^3}(h - \tilde{\delta}_\infty)^3 \qquad [2.32b]$$

where $\Delta\rho = \rho - \rho_0$. The normalizing constant for the velocity U_0 is now naturally fixed as

$$U_0 = \frac{\Delta\rho g h^2}{\mu}. \qquad [2.32c]$$

The expression [2.32b] for a developed gravitational flow of a stratified liquid in a channel is well known (e.g. Yih 1965, Graebel 1960). One obtains, solving [2.32a] with respect to δ_∞

$$\delta_{\infty 1,2} = (1 \pm \sqrt{(1 - 4Q_1)})/2 \qquad [2.33a]$$

where

$$Q_1 = \left(\frac{3Q}{G_2 - G_1}\right)^{1/3}. \qquad [2.33b]$$

We conclude from [2.33] that for any mass flux of the suspension Q into the channel such that $Q < (G_2 - G_1/192)$, there are two possible positions of the interface at large distances from the entrance. Thus the positive sign in [2.33a] corresponds to a suspension asymptotically occupying less than half of the channel width with an interface asymptotically moving down the channel (this can be easily checked by substituting $\delta_{\infty 1}$ into [2.27]), [2.29], [2.11] and [2.12].

Similarly, the negative sign in [2.23a] corresponds to a suspension asymptotically occupying more than half of the channel width with an interface moving up the channel. The first case corresponds exactly to the supercritical regime described by Probstein *et al.* (1977) and the second case to the subcritical regime of these authors. Our goal will be to describe the development of one or another of the asymptotic regimes, depending on conditions at the entrance. Since our analysis is carried out on the basis of the integral method described above, it seems appropriate to test the method on a simple, although nontrivial example, for which an independent solution is known. We point out that the only source of motion in the region of developed flow, described by the expressions [2.31], [2.27]–[2.30], [2.11]–[2.14], is the density difference between the adjacent parallel liquid columns. The character of the flow in the transition region, and the choice between the possible asymptotic behaviors is obviously determined by conditions at the entrance to the channel, in particular by the value of $\delta(0)$. (At the same time it is by no means obvious that solutions with the asymptotic limits [2.33] are realizable for any $Q < (G_1 - G_2/192)$, that is that these asymptotic limits are reachable from the entrance for any given $0 < \delta(0) < 1$.) For a homogeneous fluid an immediate analog of the situation described would be a creeping jet supplied to a part of the entrance of a channel, closed at infinity and filled with fluid (see figure 3a). It is clear that, in the absence of density stratification, the velocity disturbances caused by the jet would decay towards the closed end of the channel. This example has been used in order to test the efficiency of the integral method described above (see Appendix for details). The appropriate solution based on the above integral method predicted decomposition of a jet entering such a "dead" channel, into a sequence of decaying sing altering vortices (figures 3b). A comparison with an independent solution by Moffat (1964) has demonstrated a good agreement which encouraged us to apply the described integral method to the problem of nonhomogeneous flow under investigation. It is worth mentioning that a neglect of the transversal "viscous" pressure variation, in the example of a creeping jet, eliminated the jet's decomposition into vortices and lead to its monotonic decay. A similar neglect of the transversal pressure variations in the nonhomogeneous case is discussed in the beginning of the next section.

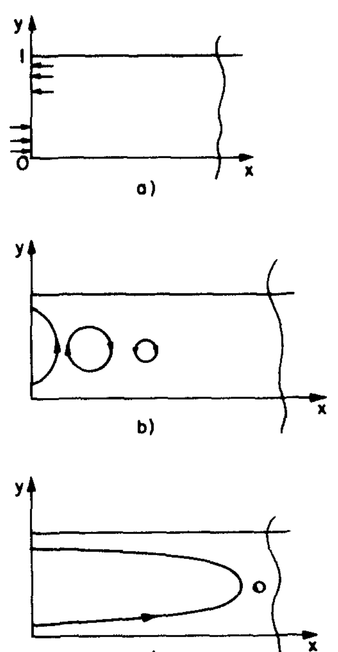


Figure 3. Jet in a "dead-ended" channel. (a) Geometry. (b) Sketch of streamlines in the creeping regime. (c) Sketch of streamlines in the inertial regime.

2.4 *Study of a gravitational stratified flow in a channel by means of the integral method*

Before we solve [2.31] numerically, let us construct a qualitatively instructive analytical solution of a simplified equation, obtained by neglecting the transverse pressure variations within the channel. It is easy to see that in this approximation the order of the equation for $\delta(x)$ reduces from fourth to second and one hàs instead of [2.31]

$$A_1{}^{II}\frac{\delta^2}{3}+B_1{}^{II}\frac{\delta}{2}+2(A_1-A_2)(1-\delta)=(G_2-G_1)(1-\delta). \qquad [2.34]$$

Upon substituting [2.27]–[2.29] into [2.34], one obtains after some transformations

$$\delta^{II}+4\delta^{I2}\frac{\delta-\frac{1}{2}}{\delta(1-\delta)}-\delta^{-1}(1-\delta)^{-1}\left(\delta-\frac{1}{2}\right)^{-1}=-\frac{G_2-G_1}{Q}\frac{\delta^2(1-\delta)^2}{\delta-\frac{1}{2}}. \qquad [2.35]$$

The substitution

$$\zeta\equiv\frac{\delta^{I2}}{2} \qquad [2.36]$$

then leads to the equation

$$\frac{d\zeta}{d\delta}+8\zeta\frac{\delta-\frac{1}{2}}{\delta(1-\delta)}=-\frac{G_2-G_1}{3Q}\frac{\delta^2(1-\delta)^2}{\delta-\frac{1}{2}}+\delta^{-1}(1-\delta)^{-1}\left(\delta-\frac{1}{2}\right)^{-1} \qquad [2.37]$$

which is linear with respect to ζ. Solving [2.37] one obtains

$$\zeta(\delta)=156\,\delta^4(1-\delta)^4\ln\frac{\left|\delta-\frac{1}{2}\right|}{\sqrt{(\delta(1-\delta))}}\left(4^3-\frac{G_2-G_1}{3Q}\right)+2\delta^3(1-\delta)^3\left(4^3-\frac{G_2-G_1}{3Q}\right)+16\delta^2(1-\delta)^2$$
$$+8/3\delta(1-\delta)+1/2+c\delta^4(1-\delta)^4 \qquad [2.38]$$

where c is a constant of integration. The solution of [2.35] is obtained from [2.37], [2.38] in implicit form as

$$x=\frac{1}{\sqrt{2}}\int_{\delta_0}^{\delta(x)}\frac{d\tau}{\pm\sqrt{(\zeta(\tau))}} \qquad [2.39]$$

where $\delta_0=\delta(0)$.

The choice between the two possible asymptotic limits for $\delta(x)$ as $x\to\infty$ becomes clear from considering [2.38]. Indeed, one sees from [2.38] that $\zeta=\delta^{I2}/2$ has a logarithmic singularity at $\delta=1/2$. This means that a solution of [2.35] with a bounded mechanical energy density cannot cross the line $\delta=1/2$ in the (δ,x) plane. (It follows from [2.13], [2.14], [2.27]–[2.30] that the non-boundness of δ' at any point means a non-bounded growth of V_1, V_2 at the same point, and this leads to a non-bounded mechanical energy density). Thus, the realizable asymptotic solution δ_∞ must be in the (x,δ) plane on the same side of the straight line $\delta=1/2$ as $\delta(0)$. The constant c in [2.38] is found from the condition $x\to\infty$ as $\delta\to\infty$. One obtains

$$c=-\frac{1}{Q_1{}^4}\left[16Q_1{}^4\ln\frac{\sqrt{(1-4Q_1)}}{2\sqrt{Q_1}}\left(4^3-\frac{1}{Q_1{}^3}\right)+2Q_1{}^3\left(4^3-\frac{1}{Q_1{}^3}\right)+16Q_1{}^2+\frac{8}{3}Q_1+\frac{1}{2}\right]. \qquad [2.40]$$

In figure 4(a) are presented typical examples of the behavior of $\zeta = 1/2(\mathrm{d}\delta/\mathrm{d}x)^2$ with δ for different values of the parameter Q_1. It is clear from figure 4(a) that the interface defined by [2.39] behaves monotonically, and the sign in [2.39] is determined uniquely by the value of $\delta(0)$ in a way that makes the phase point move towards the branch point ($\delta = \delta_\infty$, $\mathrm{d}\delta/\mathrm{d}x = 0$). Let us sketch the behavior for $\delta_0 < 1/2$ (subcritical regime). In figure 4(b) the dependence of $\mathrm{d}\delta/\mathrm{d}x$ on δ is shown for the data of figure 4(a). It is clear from figure 4(b) that the case $\delta_0 < \delta_\infty < 1/2$ corresponds to the positive sign in [2.39] whereas $\delta_\infty < \delta_0 < 1/2$ corresponds to the negative sign (the continuous lines in figure 4(b)). The corresponding behavior of the interface is shown schematically in figure 4(c). The behavior for $\delta_0 > 1/2$ (supercritical regime) is analyzed similarly.

Recall, that the expression [2.39] predicting a monotonic behavior for the interface, was obtained neglecting the transverse pressure variations. Let us remove this assumption and return to the full equation [2.31]. The equation was solved numerically, taking into account [2.27] and solutions were sought which were bounded and monotonic as $x \to \infty$, for different values of Q, $\delta(0)$. The solutions were constructed by "shooting", using a fourth-order Runge–Kutta method. Some typical results are presented in figure 5. Two qualitatively different transition regimes to the asymptotic limits are indicated: (1) a monotonic one, for the case when the initial thickness of the suspension ("clear" water) layer is greater than its asymptotic value, in the super-(sub-) critical regime; (2) a nonmonotonic regime, which becomes more pronounced as the discharge Q increases, for the case when the initial thickness of the

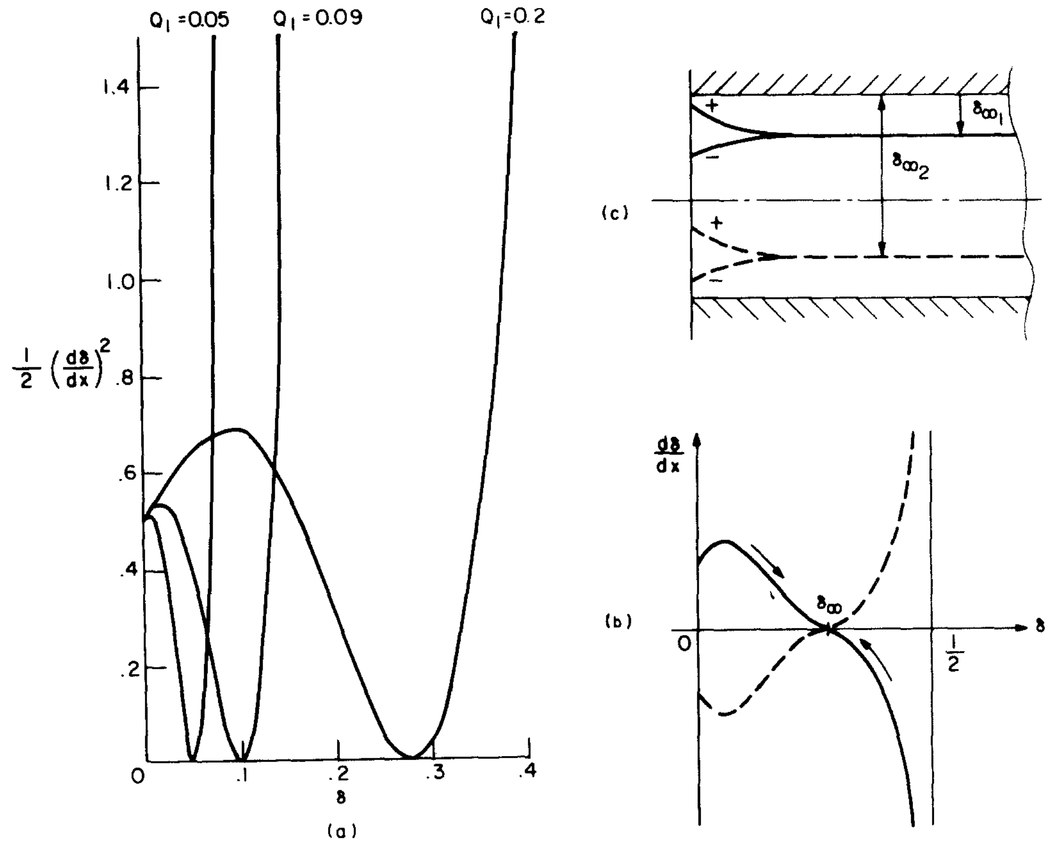
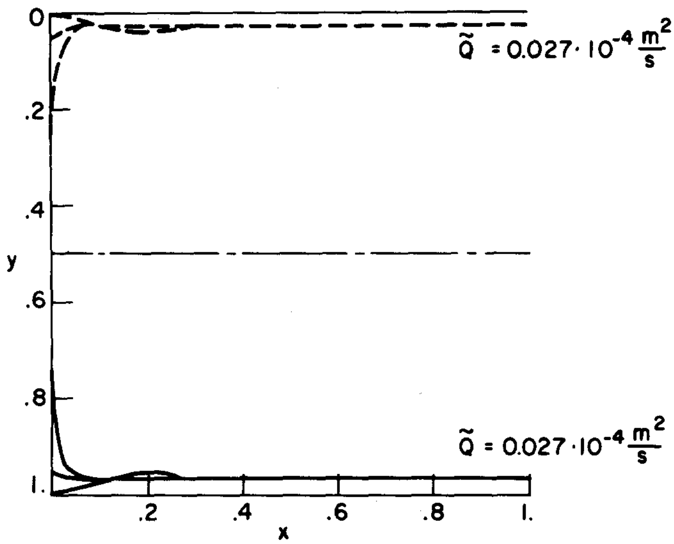


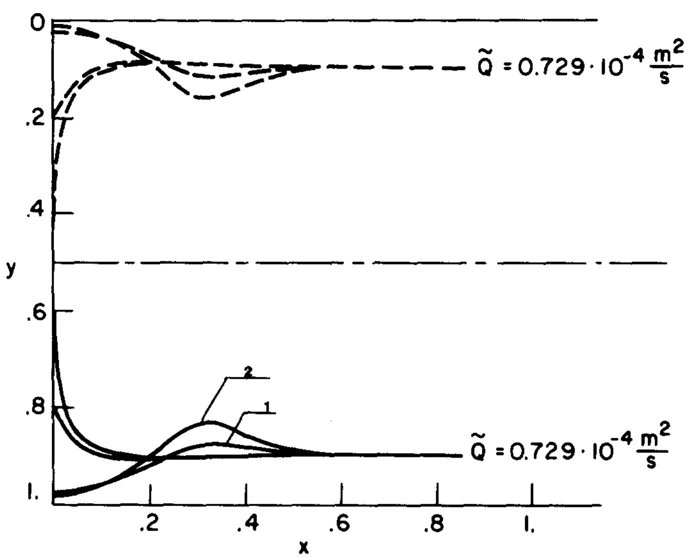
Figure 4(a). Calculated interface slopes in the creeping flow approximation, neglecting transverse pressure drop.

Figure 4(b). Sketch of interface slope in the creeping flow approximation, neglecting transverse pressure pressure drop.

Figure 4(c). Interface behavior in the creeping flow approximation, neglecting transverse pressure drop.



(a)



(b)

Figure 5. Calculated interfaces in the creeping flow approximation, $h = 10^{-2}$ m, $\Delta\rho = 0.3 \times 10^2$ kg/m^3, $\mu = 10^{-3}$ kg/m · s, $\rho = 10^3$ kg/m^3. (a) Volumetric flow $\tilde{Q} = 0.027 \cdot 10^{-4}$ m^2/s. (b) Volumetric flow $\tilde{Q} = 0.729 \cdot 10^{-4}$ m^2/s.

appropriate layer is much smaller than the corresponding asymptotic value. The observed nonmonotonicity of the interface, resulting from the transverse"viscous" pressure variations, corresponds to the formation of vortices in the case of the homogeneous creeping jet, mentioned above (see also the Apendix).

We would point out that in the course of numerical simulations at large $Q(\sim 2.5 \times 10^{-3})$, it turned out to be impossible to construct interfaces analogous to those marked 1 and 2 in figure 5(b). This was because in these cases the "crest" of the nonmonotonic interface crossed the singular surface $\delta = 1/2$ and as a result, the numerical procedure blew up.

2.5 *Inertial effects*

Here we remove the assumption of creeping flow, which was introduced in section 2.1 to simplify the problem, at the same time we retain the other principal assumptions made there.

Equations [2.1] and [2.2] with the inertial terms assume the form

$$\eta_i \nabla^2 U_i = \frac{\partial P_i}{\partial x} - G_i + r_i\left(V_i \frac{\partial U_i}{\partial y} + U_i \frac{\partial U_i}{\partial x}\right) \qquad [2.41]$$

$$\eta_i \nabla^2 V_i = \frac{\partial P_i}{\partial y} + r_i\left(V_i \frac{\partial V_i}{\partial y} + U_i \frac{\partial V_i}{\partial x}\right),\ i = 1, 2\,. \qquad [2.42a]$$

Here

$$r_i = \frac{\rho_i h U_0}{\mu_1}\,. \qquad [2.42b]$$

Following the pattern outlined in section 2.2 one obtains after some transformations and upon substituting the polynomials [2.11] and [2.12] into [2.41] and [2.42]

$$P_i(x, y) = P_i^S + \Delta p_i(x, y),\ i = 1, 2\,. \qquad [2.43]$$

Here P_1^S, P_2^S are given by the expressions [2.15], [2.16] and

$$\Delta p_1 = r_1\left[\frac{y^6}{18}(A_1A_1^{II} - A_1^{I2}) + \frac{y^4}{8}(B_1B_1^{II} - B_1^{I2}) + \frac{y^5}{5}\left(\frac{A_1B_1^{II}}{2} + \frac{A_1^{II}B_1}{3} - \frac{5}{6}A_1^IB_1^I\right)\right] \qquad [2.44a]$$

$$\Delta p_2 = r_2\left[\frac{(1-y)^6}{18}(A_2A_2^{II} - A_2^{I2}) + \frac{(1-y)^4}{8}(B_2B_2^{II} - B_2^{I2}) + \frac{(1-y)^5}{5}(A_2B_2^{II}/2 + A_2^{II}B_2/3 - \frac{5}{6}A_2^IB_2^I)\right]. \qquad [2.44b]$$

Averaging [2.41] over the thickness of each layer and taking into account [2.43] and [2.44] gives respectively,

$$\eta_1\left[A_1^{IV}\frac{\delta^4}{60} + B_1^{IV}\frac{\delta^3}{24} + \frac{2}{3}A_1^{II}\delta^2 + B_1^{II}\delta + 2A_1\right] - P_1^{0I} = -G_1 + r_1\left[\frac{\delta^4}{5}\left(\frac{B_1B_1^{III} - B_1^IB_1^{II}}{8} + \frac{A_1A_1^I}{3}\right) + \frac{\delta^3}{6}A_1^IB_1 + \frac{\delta^2}{6}B_1B_1^I + \frac{\delta^6}{126}(A_1A_1^{III} - A_1^IA_1^{II}) + \frac{\delta^5}{30}\left(\frac{A_1B_1^{III}}{2} + \frac{A_1^{III}B_1}{3} - \frac{A_1^IB_1^{II}}{3} - \frac{A_1^{II}B_1^I}{2}\right)\right] \qquad [2.45]$$

$$\eta_2\left[A_2^{IV}\frac{(1-\delta)^4}{60} + B_2^{IV}\frac{(1-\delta)^3}{24} + \frac{2}{3}A_2^{II}(1-\delta)^2 + B_2^{II}(1-\delta) + 2A_2\right] - P_2^{0I} = -G_2 - r_2\left[\frac{(1-\delta)^4}{5}\left(\frac{B_2B_2^{III} - B_2^IB_2^{II}}{8} + \frac{A_2A_2^I}{3}\right) + \frac{(1-\delta)^3}{6}A_2^IB_2 + \frac{(1-\delta)^2}{6}B_2B_2^1 + \frac{(1-\delta)^6}{126}(A_2A_2^{III} - A_2^IA_2^{II}) + \frac{(1-\delta)^5}{30}\left(\frac{A_2B_2^{III}}{2} + \frac{A_2^{III}B_2}{3} - \frac{A_2^IB_2^{II}}{3} - \frac{A_2^{II}B_2^I}{2}\right)\right]. \qquad [2.46]$$

On the other hand, substitution of [2.44] into the condition [2.7] yields

$$P_2^0 - P_1^0 = \eta_2\left[A_2^{III}\frac{(1-\delta)^4}{12} + B_2^{III}\frac{(1-\delta)^3}{6}\right] - \eta_1\left[A_1^{III}\frac{\delta^4}{12} + B_1^{III}\frac{\delta^3}{6}\right] + r_1\left[\frac{\delta^6}{18}(A_1A_1^{II} - A_1^{I2}) + \frac{\delta^4}{8}(B_1B_1^{II} - B_1^{I2}) + \frac{\delta^5}{5}\left(\frac{A_1B_1^{II}}{2} + \frac{A_1^{II}B_1}{3} - \frac{5}{6}A_1^IB_1^I\right)\right] - r_2\left[\frac{(1-\delta)^6}{18}(A_2A_2^{II} - A_2^{I2}) + \frac{(1-\delta)^4}{8}(B_2B_2^{II} - B_2^{I2}) + \frac{(1-\delta)^5}{5}\left(\frac{A_2B_2^{II}}{2} + \frac{A_2^{II}B_2}{3} - \frac{5}{6}A_2^IB_2^I\right)\right]. \qquad [2.47]$$

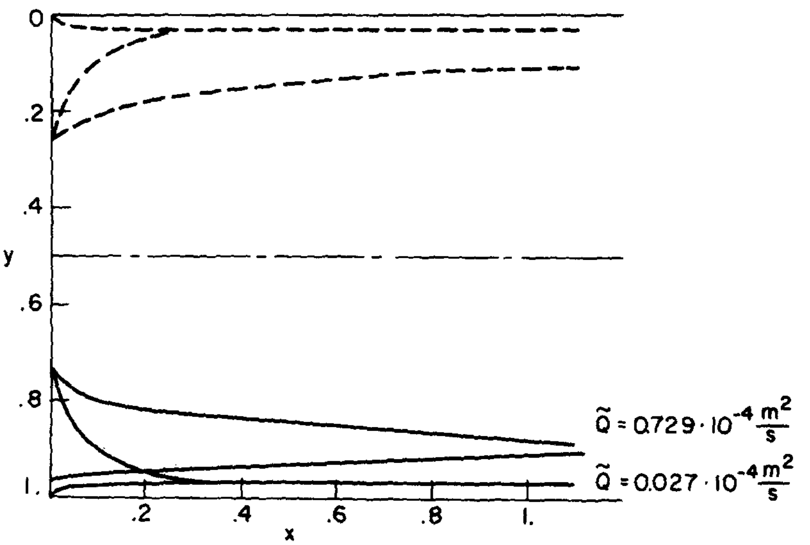


Figure 6. Calculated interfaces with inertial effects.

Equations [2.45] and [2.46] together with [2.47] and the expressions [2.47]–[2.30] define an ordinary differential equation for the unknown interface $\delta(x)$ which is the inertial counterpart of [2.31]. The system [2.45]–[2.47], [2.27]–[2.30] was solved numerically subject to the appropriate boundary conditions. Some of the numerical solutions obtained are presented in figure 6. We point out that taking into account inertial effects virtually eliminates the interface nonmonotonicities observed in the creeping approximation, and leads to an appreciable increase in the entrance length of the flow (see figure 5).

It is worth mentioning that taking into account the inertial effects in the case of a homogeneous jet in a closed channel also appreciably increases the length of the entrance region and makes the vortices (analogs of the nonmonotonicity of the interface above) virtually unobservable (see figure 3(c) and Appendix).

CONCLUSIONS

(1) A relation is established between the description of the dynamics of lamella settlers based on considering immiscible viscous fluid flow, as suggested by Probstein *et al.* (1977), and the methods of two-phase flow theory.

(2) The nature of the two alternative operating regimes of lamella settlers found by Probstein *et al.* (1977) is clarified. To this end a two-dimensional model was developed of a steady flow of viscous immiscible fluids in a channel. On the basis of this model a correspondence was established between the development of one of the alternative flow regimes and conditions at the entrance to the channel. The possibility was pointed out of a nonmonotonic interface behavior for certain entrance conditions in the regime of creeping flow. In the inertial regime the interface behaved essentially monotonically and an increase was observed in the entrance length of the flow as compared with that for the creeping regime.

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APPENDIX

2.4 *Decay of a homogeneous jet in a "dead-ended" channel*

Consider a semi-infinite channel, closed at infinity and filled with a homogeneous viscous fluid. Let a slow jet of the same fluid be introduced to a part of the open end of the channel (at the same time an appropriate amount of fluid is leaving the channel through its open end, so that the total volume flux through the channel is equal to zero). We seek the velocity distribution within the channel.

The appropriate boundary value problem may be written

$$\Delta U = \frac{\partial P}{\partial x} \qquad [A1]$$

$$\Delta V = \frac{\partial P}{\partial y} \qquad [A2]$$

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \qquad [A3]$$

$$U|_{y=0} = V|_{y=0} = U|_{y=1} = V|_{y=1} = 0 \qquad [A4]$$

$$\int_0^1 U \, \mathrm{d}y = 0 \qquad [A5]$$

where the coordinate axes are directed as indicated in figure 3(a). Recall that we are looking for solutions bounded as $x \to \infty$.

Let us apply to the solution of [A1]–[A5] the integral method described in section 2.2. The simplest polynomial approximation for $U(x, y)$ leading to $U(x, y) \neq 0$ over the entire channel will be a cubic with respect to y. Thus, we assume

$$U(x, y) = A(x)y^3 + B(x)y^2 + C(x)\, y. \qquad [A6]$$

Substitution of [A6] into [A4] and [A5] gives

$$A + B + C = 0 \qquad [A7]$$

$$\frac{A}{4} + \frac{B}{3} + \frac{C}{2} = 0. \qquad [A8]$$

The integral form of the momentum equation [A1] assumes the form, similar to [2.20],

$$\frac{A^{IV}}{120}+\frac{B^{IV}}{60}+\frac{C^{IV}}{24}+\frac{A^{II}}{2}+\frac{2}{3}B^{II}+C^{II}+3A+2B=P_0{}^I, \quad [A9]$$

where P_0 is an unknown function of x analogous to $P_1{}^0$ in [2.19].

Because of the cubic approximation, the set of equations [A7]–[A9] has to be supplemented with an integral equation for mechanical energy balance of the form

$$\int_0^1 U\left(\nabla^2 U-\frac{\partial P}{\partial x}\right)\mathrm{d}y=0. \quad [A10a]$$

Substitution of [A6] into [A10a] gives after integration and after taking into account [A7] and [A8],

$$\frac{1}{10080}A^{IV}-\frac{2}{840}A^{II}+\frac{A}{20}=0. \quad [A10b]$$

The solution of [A10b] bounded as $x\to\infty$ is of the form

$$A=A_0\,\mathrm{e}^{-K_1x}\cos(K_2x+\alpha) \quad [A11]$$

where A_0, α are integration constants, determined by the conditions at $x=0$ and where

$$K_1\simeq 4.20, \qquad K_2\simeq 2.44.$$

One concludes from [A11] that the creeping jet described here decomposes into a sequence of sign altering eddies with intensities decaying exponentially towards the closed end of the channel (a sketch of the corresponding streamlines is shown in figure 3b). The problem of a creeping jet in a closed channel was solved by Moffat (1964) by a different method. We reproduce below the data from Moffat (1964) for K_1, K_2, denoting them by a bar on top.

$$\bar{K}_1=4.21, \qquad \bar{K}_2=2.26.$$

The agreement with our approximate values given above is very close and supports the use of the integral method described for solving the system [2.1]–[2.12] for a nonhomogeneous fluid flowing in a channel.

We point out here that neglecting transverse pressure variations in [A1]–[A5] would lead to a reduction of the order of [A10b] from fourth to second, and as a result, to a monotonic decay of the jet (without decomposing into vorticies).

Finally, inclusion of the inertial terms in [A1], [A2] leads, instead of [A10b], to a nonlinear equation:

$$\frac{1}{10080}A^{IV}-\frac{2}{840}A^{II}+\frac{221}{560}AA^{I}+\frac{A}{20}=0. \quad [A12]$$

Equation [A12] was solved numerically. A scheme of a typical behaviour of the solution of [A12] for $A|_{x=0}=0(1)$ is presented in figure 3(c) and can be described as follows. The entering disturbance decays montonically, on a length scale about 20 times greater than the one in the creeping case ([A10b]), until its magnitude decreases about 100 times as compared with the entrance value after which the creeping regime, with its typical decomposition into vortices, takes over. Such a behaviour of the solution of [A12] makes vortices virtually unobservable in the inertial case. The analog of this in the nonhomogeneous case is a considerable increase in the entrance length in the inertial regime, as compared to the creeping regime and a virtual disappearance of the nonmonotonicities of the interface.